Notice how this integral was done in polar coordinates, but with z just tacked on. This is exactly what <u>cylindrical coordinates</u> are:

(r,θ,z) where $\boxed{x = r\cos\theta,\ y = r\sin\theta,\ z = z}$.

As z is just tacked on, we find

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

Cylindrical coordinates can help with triple integrals as polar did w/ double, as the previous example shows.

<u>Ex</u>: a) Write the point with cylindrical coordinates $(4,\frac{\pi}{3},-2)$ in cartesian coordinates.

b) Write the point with cartesian coordinates $(-2,2\sqrt{3},3)$ in cylindrical coordinates.

<u>Sol</u>: a) $(x,y,z) = (r\cos\theta, r\sin\theta, z) = (4\cos\frac{\pi}{3}, 4\sin\frac{\pi}{3}, -2) = (2, 2\sqrt{3}, -2)$

b) $r^2 = x^2 + y^2 = 4 + 12 = 16 \Rightarrow r = 4$

$\tan\theta = \frac{y}{x} = -\sqrt{3} \Leftrightarrow \theta = \frac{2\pi}{3} + n\pi$. Take $\theta = \frac{2\pi}{3}$ since $(-2,2\sqrt{3})$ is in quadrant II.

So, $(r,\theta,z) = (4, \frac{2\pi}{3}, 3)$

◇

15.9 - Spherical Coordinates

As we can use cylinders to give coordinates on $\mathbb{R}^3$, we can also use spheres. These coordinates are obtained by rotating polar coordinates into $\mathbb{R}^3$.

Spherical coordinates are (ρ, θ, φ) where ρ is the distance from the origin, θ is the angle made with the positive x-axis in the xy-plane, and φ is the angle made with the positive z-axis. So, we have $\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$.

The relation to cartesian is:

$$\boxed{x = \rho\cos\theta\sin\varphi, \ y = \rho\sin\theta\sin\varphi, \ z = \rho\cos\varphi}$$

We also have $\boxed{\rho^2 = x^2 + y^2 + z^2}$

Ex: a) Write the point with spherical coordinates $(3, \frac{\pi}{2}, \frac{3\pi}{4})$ in cartesian coordinates.

b) Write the point with cartesian coordinates $(-1, 1, -\sqrt{2})$ in spherical coordinates.

Sol: a) $x = \rho\cos\theta\sin\varphi = 3\cos\frac{\pi}{2}\sin\frac{3\pi}{4} = 3 \cdot 0 \cdot \frac{\sqrt{2}}{2} = 0$

$y = \rho\sin\theta\sin\varphi = 3\sin\frac{\pi}{2}\sin\frac{3\pi}{4} = 3 \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$

$z = \rho\cos\varphi = 3\cos\frac{3\pi}{4} = \frac{-3\sqrt{2}}{2}$.

$(x,y,z) = \left(0, \frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$.

b) $\rho^2 = x^2+y^2+z^2 = 1+1+2 = 4 \Rightarrow \rho = 2$

$z = \rho\cos\varphi \Leftrightarrow -\sqrt{2} = 2\cos\varphi \Rightarrow \cos\varphi = -\frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{3\pi}{4}$.

$y = \rho\sin\theta\sin\varphi \Leftrightarrow 1 = 2\sin\theta\sin\frac{3\pi}{4} = \sqrt{2}\sin\theta$

$\Rightarrow \sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Check with x:

$x = \rho\cos\theta\sin\varphi \Leftrightarrow -1 = 2\cos\theta\sin\frac{3\pi}{4} = \sqrt{2}\cos\theta$

So $\cos\theta \leq 0 \Rightarrow \theta = \frac{3\pi}{4}$

Thus, $(\rho,\theta,\varphi) = (2, \frac{3\pi}{4}, \frac{3\pi}{4})$. ◇

In spherical coordinates,

$$\boxed{dV = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}$$

Ex: Find the volume of the region inside the sphere $x^2+y^2+z^2 = 4z$ and above the cone $z = \sqrt{\frac{1}{3}(x^2+y^2)}$.

Sol: Let's rewrite the equations in spherical coords:

$\rho^2 = x^2+y^2+z^2 = 4z = 4\rho\cos\varphi$: Sphere

$\Rightarrow \boxed{\rho = 4\cos\varphi}$

$\rho\cos\varphi = z = \sqrt{\frac{1}{3}(x^2+y^2)} = \sqrt{\frac{1}{3}(\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi)} = \sqrt{\frac{1}{3}\rho^2\sin^2\varphi}$: Cone

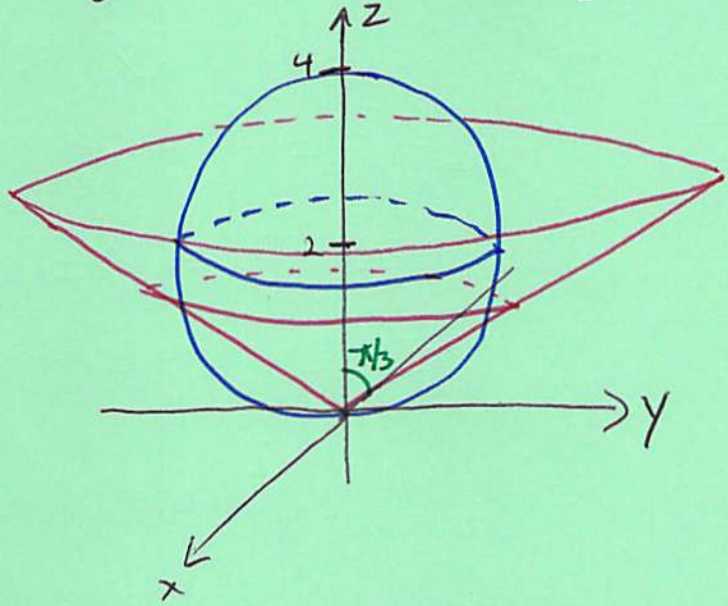
$= \frac{1}{\sqrt{3}}\rho\sin\varphi \Rightarrow \cos\varphi = \frac{1}{\sqrt{3}}\sin\varphi \Rightarrow \tan\varphi = \sqrt{3} \Rightarrow \boxed{\varphi = \frac{\pi}{3}}$

Rewriting the sphere in standard form gives:

$$x^2+y^2+(z-2)^2=4,$$

a sphere of radius 2 centered at $(0,0,2)$.

The cone is given by $\varphi=\frac{\pi}{3}$, so a sketch of the region is:



We end up with a "snocone" shape.

Then, the volume is

$$Vol=\iiint_E dV=\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{4\cos\varphi}\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$=\int_0^{\pi/3}\int_0^{2\pi}\frac{64}{3}\cos^3\varphi\sin\varphi\, d\theta\, d\varphi=\frac{128\pi}{3}\int_0^{\pi/3}\cos^3\varphi\sin\varphi\, d\varphi$$

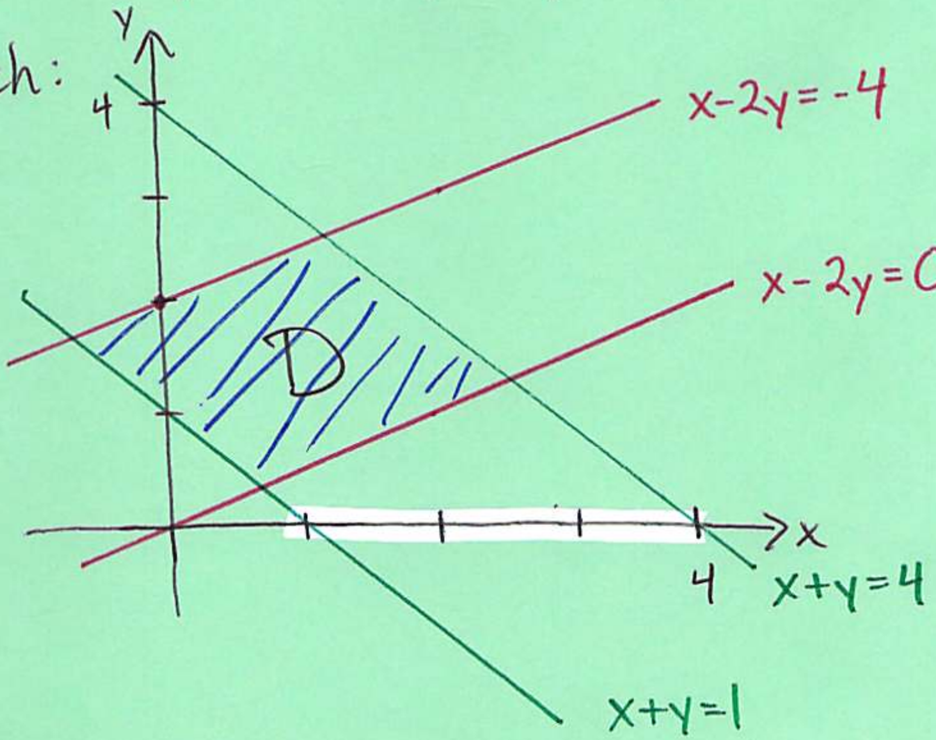
$$=\frac{32\pi}{3}\left(-\cos^4\varphi\right)\Big|_0^{\pi/3}=\frac{32\pi}{3}\left(-\frac{1}{16}-(-1)\right)=10\pi$$

$\diamond$

15.10 - Change of Variables

Ex: Compute $\iint_D 3xy\,dA$ where D is the region bounded by $x-2y=0$, $x-2y=-4$, $x+y=4$, and $x+y=1$.

Sol: Sketch:



Notice that, to do this integral would require splitting it into 3 pieces... There must be an easier way...
Notice that the opposite sides of the parallelogram are described by the same function:

$$x-2y=-4,0 \qquad x+y=1,4$$

If we write $u=x-2y$ & $v=x+y$, then the region is bounded by: $u=-4$, $u=0$, $v=1$, $v=4$ in the uv-plane... much simpler! We need to replace x&y, so we solve for them in terms of u&v:

$$\begin{cases} u = x - 2y & ① \\ v = x + y & ② \end{cases}$$

② − ① : $v - u = 3y \Rightarrow y = \frac{1}{3}(v-u)$

② $\Rightarrow x = v - y = v - \frac{1}{3}(v-u) = \frac{1}{3}(2v+u)$

So, $3xy = \frac{1}{3}(v-u)(2v+u) = \frac{1}{3}(2v^2 - uv - u^2)$.

What about dA? It is given by

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the transformation

and is given by $\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$

So, in this example:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{-1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} - \left(-\frac{2}{9}\right) = \frac{1}{3}$$

thus $dA = \frac{1}{3}\,du\,dv$.

So, the integral becomes:

$$\iint_D 3xy\,dA = \int_1^4 \int_{-4}^0 \frac{1}{9}(2v^2 - uv - u^2)\,du\,dv = \frac{164}{9}$$

◇

If we write $T(u,v) = (x(u,v), y(u,v))$ to represent the transformation, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \det DT(u,v)$$

.

<u>Def</u>: $T(u,v) = (x(u,v), y(u,v))$ is C' if its components have continuous first partials.

Change of Variables Formula (2 variables)

Suppose $T(u,v) = (x(u,v), y(u,v))$ is C' and sends the region S in the uv-plane to the region R in the xy-plane. If the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is nonzero at all points in S, $f(x,y)$ is continuous on R, and T is one-to-one on S, except maybe on the boundary of S, then:

$$\boxed{\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv}$$

$$= \iint_{T^{-1}(R)} f(T(u,v)) \left|\det DT\right| du\, dv$$

This theorem can also be used to make integrands simpler. This is more like u-substitution. Before an example of this, a neat trick from linear algebra:

$$\det(A^{-1}) = \frac{1}{\det A}$$

If $T(u,v) = (x(u,v), y(u,v))$, then $T^{-1}(x,y) = (u(x,y), v(x,y))$.

So, $\dfrac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1})$. But $D(T^{-1}) = DT^{-1}$.

Thus $\dfrac{\partial(u,v)}{\partial(x,y)} = \det D(T^{-1}) = \det DT^{-1} = \dfrac{1}{\det DT} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$

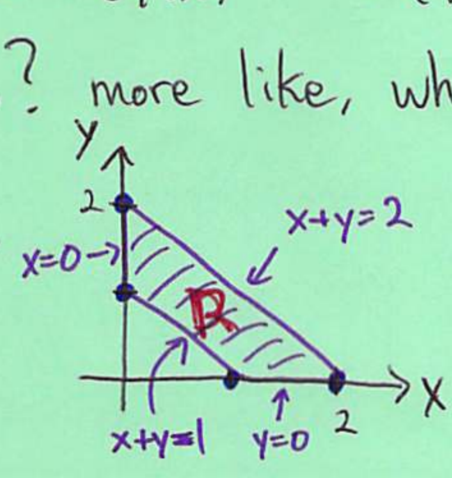
so, $\boxed{\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{\frac{\partial(u,v)}{\partial(x,y)}}}$

Ex: Compute $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$ where R is the trapezoidal region with vertices $(1,0), (2,0), (0,2), (0,1)$.

Sol: $\cos\left(\frac{y-x}{y+x}\right)$ looks pretty hard to integrate...

If we write $u = y-x$ and $v = y+x$, then we get $\cos\left(\frac{y-x}{y+x}\right) = \cos\left(\frac{u}{v}\right)$, a bit better. What happens to R? more like, what maps to R? First, let's sketch

R:

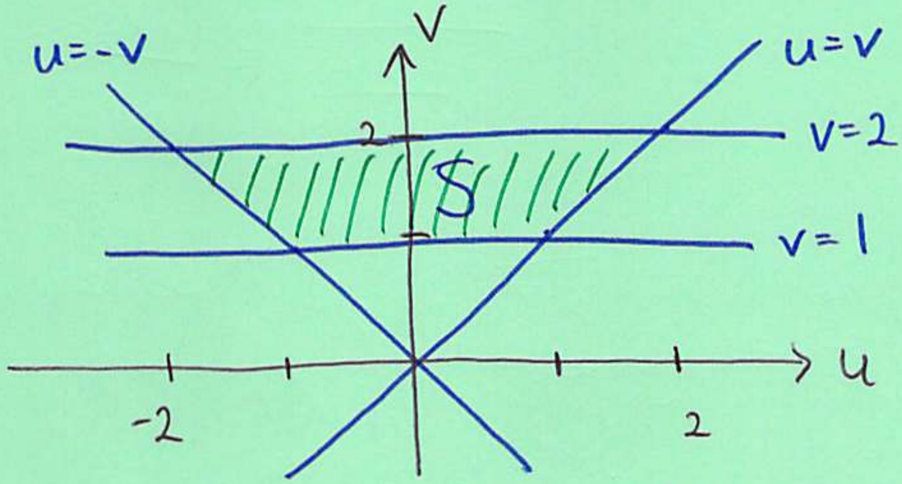


What is the region S which maps to R?

xy-plane	uv-plane
$x+y=2$	$v=2$
$x+y=1$	$v=1$
$x=0$	$\left\{\begin{array}{l} u=y-x=y \\ v=y+x=y \end{array}\right\} \Rightarrow u=v$
$y=0$	$\left\{\begin{array}{l} u=y-x=-x \\ v=y+x=x \end{array}\right\} \Rightarrow u=-v$

So, S is bounded by $v=2, v=1, u=v, \& \ u=-v$.

<u>S:</u>



Now, we need the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. Use the trick:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1=-2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2}$$

Thus, $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) \left|\frac{-1}{2}\right| du\,dv$

$$= \frac{1}{2}\int_1^2\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\,dv = \frac{3}{2}\sin(1).$$

$\Diamond$

There is a corresponding 3-variable version of the theorem. If $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ and $T(S)=R$, then

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$